Cell 2-Representations & Categorification at Prime Roots of Unity

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Abstract

For q a root of unity of prime order p, Khovanov–Qi, Elias–Qi categorified small quantum groups of type A_1 using categories enriched with a p-differential ∂.

We extend the theory of cell 2-representations of Mazorchuk–Miemietz to a p-dg enriched setup, and build on some of their results. As an application, we consider cyclotomic quotients of categorified small quantum groups.

Background

Crane–Frenkel vision (1994): Replace small quantum groups $\mathrm{u}_q(\mathfrak{g})$ by categories, study their categorical representations.

Prime Root of Unity Categorification

Assume q is a primitive p-th root of unity, p a prime.

- → $\mathrm{u}_q(\mathfrak{g})$ is an $\mathbb{O}_n$-algebra — for $\mathbb{O}_n$ the cyclotomic integers.
- → Khovanov (2005) suggested to use the **stable category** of $\Bbbk[\partial]/(\partial^p)$-modules to categorify $\mathbb{O}_p$ and modules over it.
- → Hopfological Algebra studies categories enriched in H-mod, H a Hopf algebra, to categorify interesting algebras.
- → This led to recent categorifications, at prime root of unity, of $\dot{\mathrm{u}}_q(\mathfrak{sl}_2^+)$ [Khovanov–Qi], $\dot{\mathrm{u}}_q(\mathfrak{sl}_2)$, $\dot{\mathrm{U}}_q(\mathfrak{sl}_2)$ [Elias–Qi].

2-Representation Theory

Categorified representations: 2-representations appear in the literature in Rouquier's work on 2-Kac Moody algebras (2004, 2008) and elsewhere. A systematic study started in a series of papers by Mazorchuk–Miemietz (beginning in 2010).

Think of *representation* of an algebra as a $\Bbbk$-linear functor $A \longrightarrow \mathrm{Vect}_{\Bbbk}$. Similarly, a **2-representation** is a strict $\Bbbk$-linear 2-functor $\mathbf{M}\colon \mathscr{C} \longrightarrow \mathfrak{R}_{\Bbbk}$. The target has objects which are categories equivalent to A-proj for A finite-dimensional, 1-morphisms are equivalent to functors of tensoring with projective bimodules, and 2-morphisms are morphisms of bimodules.

Cell 2-representations give a way to construct *simple transitive* 2-representations. These act transitivity on objects, with no non-trivial ideals in $\mathbf{M}$ stable under the action.

Construction: Define a partial order on 1-morphisms:

$$\mathrm{F} \leq_L \mathrm{G} \iff \mathrm{G} \text{ is a direct summand of } \mathrm{H}\circ\mathrm{F}, \text{ for some } \mathrm{H}$$

- → Equivalence classes $\mathcal{L} \subset \mathscr{C}(\mathtt{i},-)$ are called left cells of $\mathscr{C}$.
- → Principal 2-representations $\mathbf{P}_{\mathtt{i}}\colon \mathtt{j} \mapsto \mathscr{C}(\mathtt{i},\mathtt{j})$, $\quad \mathrm{G} \mapsto \mathrm{G}\circ(-)$.
- → Restict to the 2-subrepresentation $\mathbf{R}_{\mathcal{L}} \leq \mathbf{P}_{\mathtt{i}}$ generated by 1-morphism in the cell $\mathcal{L}$.
- → Cell 2-representation: the maximal quotient $\mathbf{C}_{\mathcal{L}} := \mathbf{R}_{\mathcal{L}}/\mathbf{I}$ not annihilating identities in $\mathcal{L}$.

p-DG 2-Representations

Definitions and Results

Technical restriction: All p-dg categories $\mathscr{C}(\mathtt{i},\mathtt{j})$, and $\mathbf{M}(\mathtt{i})$, are *strongly finitary*. This means:

- → The underlying $\Bbbk$-linear category is finitary and Karoubian.
- → All subquotient idempotents (not necessary annihilated by ∂) split in the enriched category.
- → All objects are filtered by $\Bbbk$-indecomposables and cofibrant (cf. *fantastic filtration* of Elias–Qi).

A ***p*-dg 2-representation** is a p-dg 2-functor $\mathscr{C} \longrightarrow \mathfrak{M}_p$, with target $\mathfrak{M}_p$ consisting of:

- → Objects that are small p-dg categories $\overline{\mathcal{A}}$, a combinatorial model for compact semi-free modules, generalizing *one-sided twisted complexes* of Bondal–Kapranov.
- → 1-Morphisms are p-dg functors.
- → 2-Morphisms are natural transformations (enriched).

We can define:

- → An analogue of principal 2-representations $\mathbf{P}_{\mathtt{i}}$, where $\mathbf{P}_{\mathtt{i}}(\mathtt{j}) = \overline{\mathscr{C}(\mathtt{i},\mathtt{j})}$.
- → An analogue of cell 2-representations $\mathbf{C}_{\mathcal{L}}$.

Theorem 1.
The p-dg cell 2-representations are simple transitive p-dg 2-representations.

The underlying additive 2-representations are inflations of the cell 2-representations by a local algebra (Mazorchuk–Miemietz).

Theorem 2.
A p-dg 2-representation $\mathbf{M}\colon \mathscr{C} \to \mathfrak{M}_p$ induces a triangulated 2-representation $\mathbf{KM}$ of the triangulated 2-category $\mathscr{K}(\mathscr{C})$ — obtained by taking stable categories of $\mathscr{C}(\mathtt{i},\mathtt{j})$.

Class of examples.
The p-dg 2-categories $\mathscr{C}_{\mathcal{A}}$, where $\mathcal{A} = \prod_{i=1}^{n} \mathcal{A}_i$ is a list of strongly finitary p-dg categories.

- → Objects $\mathtt{i}$ correspond to $\overline{\mathcal{A}}_i$.
- → 1-Morphisms are generated by tensoring with cofibrant (and $\Bbbk$-indecomposable) $\mathcal{A}_i$-$\mathcal{A}_j$-bimodules.
- → 2-Morphisms are morphisms of bimodules.

Theorem 3.
For $\mathcal{A}$ strongly finitary assume $\partial(\mathrm{rad}\mathcal{A}) \subset \mathrm{rad}\mathcal{A}$. Then for $\mathscr{C}_{\mathcal{A}}$, the cell 2-representation of the unique non-identity cell is equivalent to the natural (defining) 2-representation.

Applications to Categorified Quantum Groups

Let $0 \leq \lambda \leq p-1$. Denote by $\mathbf{L}_\lambda\colon \mathscr{U} \longrightarrow \mathfrak{R}_{\Bbbk}$ the categorification of the simple $\dot{\mathrm{u}}_q(\mathfrak{sl}_2)$-module $L(\lambda)$ of Elias–Qi. Then $\mathscr{U}^{\lambda} := \mathscr{U}/\ker \mathbf{L}_\lambda$ is strongly finitary. Fix a lowest left cell $\mathcal{L}$.

Corollary.
The p-dg 2-categories $\mathscr{U}^{\lambda}_{\mathcal{L}}$ and $\mathscr{C}_{\mathcal{A}}$ are p-dg biequivalent, where $\mathcal{A}$ is generated by regular p-dg bimodules over nil-Hecke algebras. The idempotent completion $\widehat{\mathscr{U}^{\lambda}_{\mathcal{L}}}$ is biequivalent to $\mathscr{C}_{\mathcal{B}}$, where $\mathcal{B}$ is the list of coinvariant algebras.

The 2-cell representation $\mathbf{C}_{\mathcal{L}}$ of $\widehat{\mathscr{U}^{\lambda}}$ is given by the natural action on coinvariant algebras and also categorifies $L(\lambda)$.

Theorem 4.
Every endofunctor of the categorified simple representation $\mathbf{L}_\lambda$ of $\mathscr{U}^{\lambda}$ is p-dg equivalent to an extension of the identity functor.